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# 1 THE BASICS

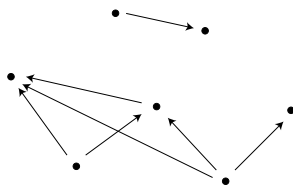
## 1.1 CATEGORIES

**Definition 1.** A category  $\mathbf{C}$  is a collection of

- **objects**  $\text{ob}(\mathbf{C})$ ; and
- **morphisms**  $\text{mor}(\mathbf{C})$ .

Let  $\text{Hom}(A, B)$  denote the morphisms from object  $A$  to object  $B$ . There are several requirements:

1. Morphisms must compose:  $(f, g) \mapsto gf$ .
2. Morphism composition is associative.
3. If  $A \neq C$  or  $B \neq D$ , then  $\text{Hom}(A, B)$  and  $\text{Hom}(C, D)$  are disjoint.
4. Each object has an identity morphism, which is a two-sided identity.



A category is **concrete** if, informally, its objects are underlying sets and its morphisms are functions between them, e.g. **Set**, **Top**, **Grp**. By contrast, **abstract** categories don't have this structure, e.g.  $BG$  for a group  $G$ .

A category is **discrete** if all its morphisms are identities, i.e. all its objects are isolated.

Because of set-theoretical issues, it's useful to denote when a category is "small enough". We say a category is **small** if it has only a set's worth of morphisms. Since

$$\text{identity morphisms} \leftrightarrow \text{objects},$$

small categories also have a set's worth of objects. We can loosen this somewhat: if  $\text{Hom}(X, Y)$  is always a set, the category is **locally small**.

**Proposition 1.** Identity morphisms and morphism inverses are unique.

**Definition 2.** An **isomorphism** is an invertible morphism.

$$X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{f^{-1}} \end{array} Y$$

Isomorphisms (isos) generalize bijective functions, which are both injective and surjective. Injective functions generalize to monomorphisms (monos), and surjective functions to epimorphisms (epis).

**Include split monos/epis.**

**Definition 3.** A morphism  $f$  is a **monomorphism** if for all parallel (between same objects) morphisms  $g, h$  with the proper domains,

$$fg = fh \implies g = h.$$

Similarly,  $f$  is an **epimorphism** if

$$gf = hf \implies g = h.$$

There's some fun vocab and symbols to go along with these. Monos are monic and denoted by  $\rightarrow$ , and epis are epic and denoted by  $\twoheadrightarrow$ . An isomorphism is necessarily both monic and epic, although the converse doesn't hold in general.

Special types of morphisms get their own special names sometimes too. An **endomorphism** is a morphism  $X \rightarrow X$ . An isomorphic endomorphism is called an **automorphism**.

**Definition 4.** A category  $\mathbf{S}$  is a **subcategory** of  $\mathbf{C}$  if

1.  $\text{ob}(\mathbf{S})$  is a subcollection of  $\text{ob}(\mathbf{C})$ ; and
2. for all  $A, B \in \text{ob}(\mathbf{S})$ ,  $\text{Hom}_{\mathbf{S}}(A, B)$  is a subcollection of  $\text{Hom}_{\mathbf{C}}(A, B)$  with identity.

A **full** subcategory doesn't remove any morphisms between the remaining objects, i.e.

$$\text{Hom}_{\mathbf{S}}(A, B) = \text{Hom}_{\mathbf{C}}(A, B).$$

**Definition 5.** A **groupoid** is a category whose morphisms are all isomorphisms.

Every category contains a subcategory called the **maximal groupoid**, which is all of the objects along with only the morphisms that are isomorphisms.

**Example 1.** We can define a **group** as a groupoid that has only one object. The group elements are the morphisms. The properties of a group follow from the properties of categories and the fact that our morphisms are all isomorphisms. Given a group  $G$ , its representation as a single-object category is denoted  $BG$ .

## 1.2 DUALITY

**Definition 6.** Given a category  $\mathbf{C}$ , its **opposite** or **dual** category  $\mathbf{C}^{\text{op}}$  is the category gotten by “reversing” the morphisms of  $\mathbf{C}$ . This means

$$\begin{aligned}\text{ob}(\mathbf{C}^{\text{op}}) &= \text{ob}(\mathbf{C}), \\ \text{Hom}_{\mathbf{C}^{\text{op}}}(A, B) &= \text{Hom}_{\mathbf{C}}(B, A).\end{aligned}$$

My biggest misconception of this at first was that we were actually reversing each morphism, but this is clearly impossible. For example, if we’re working in **Set**, we physically can’t reverse all the morphisms since not all functions are invertible.

**Note 1.** We aren’t actually changing any of the morphisms. The “reversal” of a morphism is a completely formal process. In fact, we can’t even compare  $f$  and  $f^{\text{op}}$  since they live in different categories! At the end of the day, a category’s dual has the same information, but the notation is just all reversed.

Note that the identities in a category and its dual are the same. Compositions, on the other hand, are necessarily reversed:

$$f^{\text{op}}g^{\text{op}} \doteq (gf)^{\text{op}}.$$

Duals are important because they make universal quantifications twice as valuable: if a theorem applies “for all categories”, then it certainly applies to the opposites of all categories. We can then reinterpret the theorem in the opposite case to get a dual theorem, and to prove it we just reverse all the morphisms in our original proof.

### 1.3 FUNCTORS

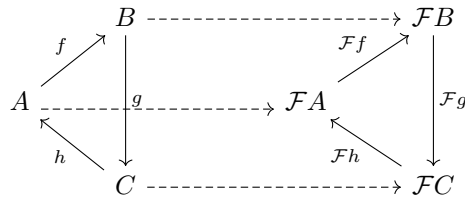
Functors are the morphisms associated with categories: they map categories to categories in ways that respect categorical structure.

**Definition 7.** A (covariant) functor  $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{D}$  satisfies:

- If  $A \in \mathbf{C}$ , then  $\mathcal{F}A \in \mathbf{D}$ .
- If  $f : A \rightarrow B$ , then  $\mathcal{F}f : \mathcal{F}A \rightarrow \mathcal{F}B$ .

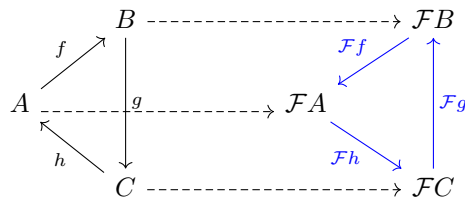
These are subject to the functoriality axioms:

- $\mathcal{F}(fg) = \mathcal{F}f \cdot \mathcal{F}g$  for all  $f, g$ .
- $\mathcal{F}1_A = 1_{\mathcal{F}A}$  for all  $A$ .



A **contravariant functor** is the same but with the morphisms  $\mathcal{F}f$  reversed. This is just a covariant functor in disguise, though: we can represent it by a covariant functor with domain  $\mathbf{C}^{\text{op}}$ .

$$\mathcal{F} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}.$$



**Example 2.** Some functors :)

1. Forgetful functors.
2.  $\mathbf{Top} \rightarrow \mathbf{Htpy}$  is the identity on objects (topological spaces) and sends morphisms (continuous functions) to their homotopy class.
3.  $\pi_1$  is a functor  $\mathbf{Top}_* \rightarrow \mathbf{Grp}$ .

**Proposition 2.** Functors preserve isos and split monos/epis.

**Definition 8.** A functor  $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{D}$  is **faithful** if for all objects  $A, B$  of  $\mathbf{C}$ , the map

$$\begin{aligned} \text{Hom}(A, B) &\rightarrow \text{Hom}(\mathcal{F}A, \mathcal{F}B) \\ f &\mapsto \mathcal{F}f \end{aligned}$$

is one-to-one.  $\mathcal{F}$  is **full** if this map is onto.

Note that the fixed  $A$  and  $B$  above are important. The injective/surjective conditions don't apply to arbitrary morphisms in  $\mathbf{C}$  since they might connect different objects.

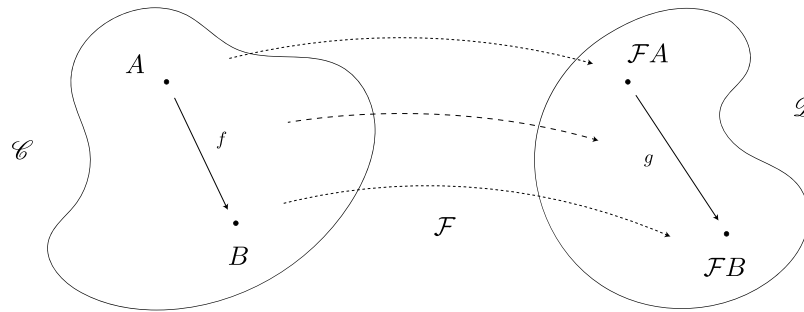
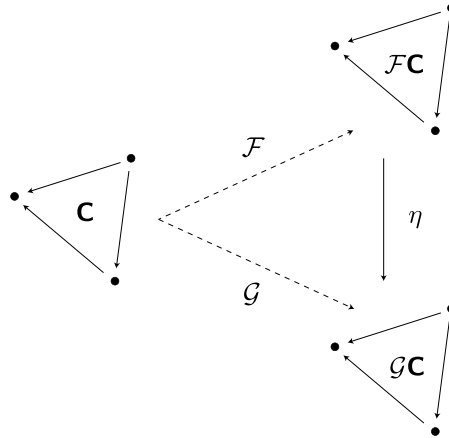


Figure 1.1: For all  $A, B$ , and  $g$ , a faithful functor sends at *most* one solid arrow in  $\mathbf{C}$  to  $g$ . A full functor sends at *least* one solid arrow in  $\mathbf{C}$  to  $g$ .

**Example 3.** The inclusion functor from  $\mathbf{S}$  to  $\mathbf{C}$  is always faithful, and it's full if and only if  $\mathbf{S}$  is a full subcategory.

## 1.4 NATURAL TRANSFORMATIONS

Natural transformations change one functor into another in a way that respects the underlying structure of the categories involved. It's kinda like a homotopy between  $\mathcal{F}$  and  $\mathcal{G}$  in the sense that for all  $C \in \mathbf{C}$ , it gives a morphism from  $\mathcal{F}C$  to  $\mathcal{G}C$ .



**Definition 9.** Suppose  $\mathcal{F}, \mathcal{G} : \mathbf{C} \rightarrow \mathbf{D}$  are functors. Then a **natural transformation**  $\alpha : \mathcal{F} \Rightarrow \mathcal{G}$  is a family of **components**

$$\{\eta_X : \mathcal{F}X \rightarrow \mathcal{G}X\}_X$$

such that the following diagram commutes for any  $f : X \rightarrow Y$  in  $\mathbf{C}$ .

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\eta_X} & \mathcal{G}X \\ \mathcal{F}f \downarrow & & \downarrow \mathcal{G}f \\ \mathcal{F}Y & \xrightarrow{\eta_Y} & \mathcal{G}Y \end{array}$$

If every  $\eta_X$  is an isomorphism, then  $\eta$  is a **natural isomorphism** and we write  $\eta : \mathcal{F} \cong \mathcal{G}$ .

## 2 UNIVERSAL PROPERTIES

### 2.1 COMMON EXAMPLES

**Definition 10.**  $(X, \{\pi_\alpha\}_\alpha)$  is a **product** of  $\{X_\alpha\}_\alpha$  if for all  $Y$  and morphisms  $f_\alpha : Y \rightarrow X_\alpha$ , there is a unique morphism  $f : Y \rightarrow X$  lifting each  $f_\alpha$ .

$$\begin{array}{ccc}
 & X & \\
 \exists! f \nearrow & & \downarrow \pi_\alpha \\
 Y & \xrightarrow{f_\alpha} & X_\alpha
 \end{array}$$

**Definition 11.**  $(X, \{i_\alpha\}_\alpha)$  is a **coproduct** of  $\{X_\alpha\}_\alpha$  if for all  $Y$  and morphisms  $f_\alpha : X_\alpha \rightarrow Y$ , there is a unique morphism  $f : X \rightarrow Y$  extending each  $f_\alpha$ .

$$\begin{array}{ccc}
 & X & \\
 \exists! f \nwarrow & & \uparrow i_\alpha \\
 Y & \xleftarrow{f_\alpha} & X_\alpha
 \end{array}$$

**Proposition 3.** If  $(X, \{\pi_\alpha\}_\alpha)$  is a product, then each  $\pi_\alpha$  is epic. If  $(X, \{i_\alpha\}_\alpha)$  is a coproduct, then each  $i_\alpha$  is monic.

**Definition 12.**  $(F, i)$  is free on the set  $B$  if for all objects  $X$  and maps  $f : B \rightarrow X$ , there is a unique morphism  $F \rightarrow X$  extending  $f$ .

$$\begin{array}{ccc}
 F & & \\
 \uparrow i & \searrow \exists! & \\
 B & \xrightarrow{f} & X
 \end{array}$$