CONTENTS

1 BASICS

variation, quadratic variation
filtration

1.1 CONTINUITY

Definition 1. A function $f : I \to \mathbb{R}$ is γ-**Hölder continuous** if there is a $C < \infty$ such that

$$|f(t) - f(s)| \leq C\,|t - s|^{\gamma}$$

for all $s, t \in I$. Functions with $\gamma = 1$ are **Lipschitz continuous**.

Theorem 1 (Kolmogorov Continuity Theorem)**.** Let $\{X_t\}$ be a stochastic process on $[0, 1]$. If there are $\alpha, \beta, C > 0$ such that

$$\mathbb{E}\left(|X_t - X_s|^{\alpha}\right) \leq C|t - s|^{1+\beta},$$

then there is a version $\tilde{X}_t$ of X_t with sample paths that are almost surely γ-Hölder continuous for $\gamma \in (0, \beta/\alpha)$.

version means $\mathbb{P}\left(\tilde{X}_t = X_t\right) = 1$ **for all** t**.**

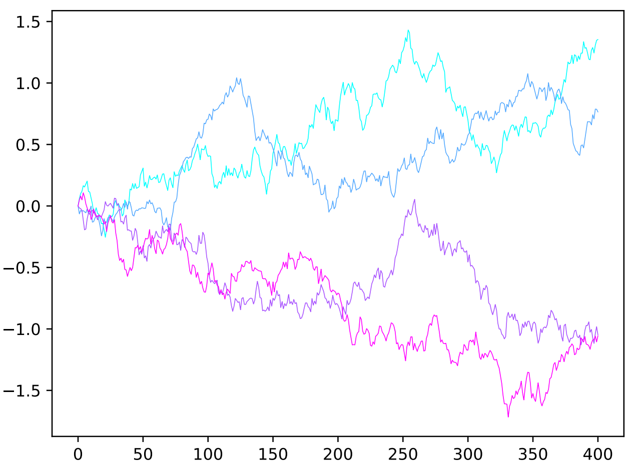
2 BROWNIAN MOTION

Definition 2. A standard **Brownian motion** $B(t, \omega)$ is a continuous time $\mathbb{R}$-valued stochastic process over some $(\Omega, \mathcal{F}, \mathbb{P})$ such that

1. $B_t - B_s \sim \mathcal{N}(0, t-s)$;
2. Disjoint increments are independent;
3. The sample path $t \mapsto B_t(\omega)$ is continuous with probability 1.

At all times, a Brownian motion receives an infinitesimal Gaussian kick. The intuition here is that "dB" is then a Gaussian random variable. Of course, dB is meaningless right now since B is nowhere differentiable with probability 1, but we will give it meaning later in terms of Itô integrals, and the interpretation will be the same.

A useful fact for proving that disjoint intervals are independent: two Gaussians are independent $\iff$ they have 0 covariance.



Proposition 1. If B_t is a Brownian motion, then so are the following two processes:

- $X_t := \frac{1}{\sqrt{\alpha}} B_{\alpha t}$ for fixed $\alpha > 0$;

- $Y_t := B_{s+t} - B_s$ for fixed $s > 0$;
- $Z_t := tB_{1/t}$.

Proposition 2. If B_t is a Brownian motion, then $\text{Cov}(B_t, B_s) = \min(t, s)$.

Construct a BM using Wiener measure and $B_t(\omega) = \omega_t$.
Is the following the finite dimensional distribution stuff?
Let $A := \{\omega \mid B_{t_k}(\omega) \in (a_k, b_k) \text{ for } k = 1, \dots, N\}$. If

$$\phi(s, y) := \frac{\exp\left(-y^2/(2s)\right)}{\sqrt{2\pi s^2}},$$

then the probability of A is

$$\mathbb{P}(A) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \phi(t_1, x_1) \prod_{i=2}^{N} \phi(t_i - t_{i-1}, x_i - x_{i-1})\, dx_1 \ \cdots \ dx_n.$$

The idea here is that $\phi(t_i - t_{i-1}, x_i - x_{i-1})$ is the conditional density for B_{t_k} given $B_{t_{k-1}} = x_{k-1}$.

Proposition 3. The sample paths of Brownian motion are almost surely γ-Hölder continuous for $\gamma \in (0, 1/2)$.

Proposition 4. If B is a Brownian motion on $[0, T]$, then with probability 1,

- $V^p(B, [0, T]) < \infty$ for $p > 2$;
- $V^p(B, [0, T]) = \infty$ for $p < 2$.

The quadratic variation of B is $[B, B](t) = t$.

3 INTEGRATION

3.1 INTEGRATION OF SIMPLE PROCESSES

Suppose B_t is a Brownian motion adapted to $\{\mathcal{F}_t\}$. Then $\mathcal{L}^2_A([0,T] \times \Omega)$ is the space of all processes $X(t,\omega)$ adapted to $\{\mathcal{F}_t\}$ such that

$$\mathbb{E}\left(\int_0^T X^2\, ds\right) < \infty.$$

This space is Banach space (complete normed vector space) with norm

$$\|X\|_{\mathcal{L}^2_A} = \sqrt{\mathbb{E}\left(\int_0^T X^2\, ds\right)}.$$

The subspace $\mathcal{L}^2_{A,0} \subset \mathcal{L}^2_A$ of *simple* adapted processes is dense in $\mathcal{L}^2_A$: for any $X \in \mathcal{L}^2_A$, there is a sequence $\{X_n\} \subset \mathcal{L}^2_{A,0}$ converging to X in the $\mathcal{L}^2$ sense, i.e.

$$\lim_{n\to\infty} \|X_n - X\|_{\mathcal{L}^2_A} = \lim_{n\to\infty} \sqrt{\mathbb{E}\left(\int_0^T (X_n - X)^2\, ds\right)} = 0.$$

We'll define the Itô integral for simple adapted processes, then extend it to general adapted processes in the next section.

finish

Proposition 5. Let $\sigma \in \mathcal{L}^2_A$, then the quadratic variation of $X(t) = \int_0^t \sigma\, dB$ is

$$[X,X](t) = \int_0^t \sigma^2\, ds.$$

Note that if σ depends on ω, then $[X,X](t)$ is still a random variable.

3.2 EXTENDING THE ITÔ INTEGRAL

For $X \in \mathcal{L}^2_A$, we know there's a sequence $\{X_n\}$ converging to X in the $\mathcal{L}^2$ sense. Then by the Itô isometry, the sequence $\{I_n\}$ given by

$$I_n := \int_0^T X_n \, dB$$

is a Cauchy sequence. Thus there is a random variable $I \in L^2$ such that $I_n \to I$ in the L^2 sense, i.e.

$$\lim_{n\to\infty} \|I_n - I\|_{L^2} = \lim_{n\to\infty} \mathbb{E}\left(|I_n - I|^2\right) = 0.$$

Definition 3. For $X \in \mathcal{L}^2_A$,

$$\int_0^T X \, dB$$

is the unique limit of the sequence given by $I_n := \int_0^T X_n \, dB$, where $X_n \to X$.

This integral has all the same properties as the one for simple processes. **Further extension where martingale property becomes *local* martingal property.**

3.3 ITÔ PROCESSES

do this.

3.4 ITÔ'S FORMULA

Theorem 2. Let Z be an Itô process satisfying

$$dZ = \mu \, dt + \sigma \, dB.$$

Let $f(t, x) \in C^2$, then

$$df(t, Z_t) = \left[\frac{\partial f}{\partial x}\right] dZ + \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\right] dt,$$

where all partial derivatives are evaluated at (t, Z_t).