STOKES' THEOREM ON MANIFOLDS

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Math 323: Geometry

1 INTRODUCTION

In any introductory course on multivariable calculus, one will encounter standard theorems like Green's and Stokes' Theorems. Although the notation for each of these theorems is somewhat intimidating upon first glance, the general concept is the same in both: just as in the fundamental theorem of calculus, to find out how small changes in a function build up over an area, it's actually enough to analyze how the function acts on the boundary of that area.

In the 1-dimensional case (the fundamental theorem of calculus), this means $f(b) - f(a)$ can be found by integrating df over $[a, b]$. Note here that $\{-a, b\}$ is the oriented boundary of the interval $[a, b]$, so we can rewrite this theorem as

$$\int_{[a,b]} df = \int_{\partial[a,b]} f.$$

The situation in higher dimensions is similar, except we have to work with different notions of the derivative that take into account how much f curls around each point. In 2 dimensions, this is known as Green's Theorem, and in 3 dimensions, it's the classical Stokes' Theorem. In fact, this relationship generalizes to arbitrary orientable manifolds. In the following sections, we will build up the theory of manifolds, tensors, and differential forms necessary to rigorously understand the fully generalized Stokes' Theorem.

2 MANIFOLDS

To generalize Stokes' Theorem to manifolds, we'll first have to define what a manifold is. Informally speaking, a manifold is any space that looks Euclidean if you zoom in far enough, with a few extra technical conditions that make them nicer to work with.

Definition 1. *A topological space M is an n-dimensional* ***manifold*** *if it is second countable, Hausdorff, and locally homeomorphic to $\mathbb{R}^n$. It is a* ***manifold with boundary*** *if some points have neighborhoods homeomorphic to the n-dimensional upper half plane $\mathbb{H}^n := \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$.*

We can then represent a manifold M as a collection of **charts** $\{(U_i, \phi_i)\}$, where each U_i is an open subset of M and each ϕ_i is a homeomorphism from U to an open subset of either $\mathbb{R}^n$ or $\mathbb{H}^n$.

Since we want to generalize the notion of the derivative, we'll also need to define when maps between manifolds are smooth. We say a map $f : U \to V$ between open subsets of $\mathbb{R}^n$

and $\mathbb{R}^m$ is **smooth** if each of the component functions of f is infinitely differentiable. To extend this to manifolds, suppose $(U, \phi) \subset M$ and $(V, \psi) \subset N$ are charts of manifolds. Then we say that a map $g : U \to V$ is smooth if $\psi \circ g \circ \phi^{-1}$ is smooth in the Euclidean sense.

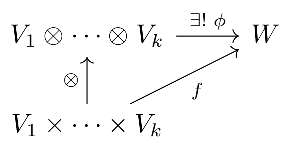
One final condition we need to ensure that our maps are well-defined is for the smooth structures induced by overlapping maps to be compatible. We say that two charts (U, ϕ) and (V, ψ) are **smoothly compatible** if either $U \cap V = \varnothing$ or $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

3 TENSORS

Although it might seem like a serious digression from the geometry, we'll need to build up the idea of tensors and tensor products: in the next section, we'll use tensor products to define the wedge product of differential forms.

Suppose we have a k-multilinear map $f : V \times \cdots \times V \to W$; we call f a k**-tensor** on V. If f had been linear, we would be able to analyze it easily using any of the theorems of linear algebra. Luckily for us, it is possible to find a space in which f actually *is* linear – this space is called the *tensor product* $V \otimes \cdots \otimes V$. We give a slightly more general definition below.

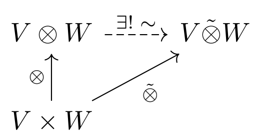
Definition 2. *The* ***tensor product*** *of* $V_1 \times \cdots \times V_k$ *is a vector space* $V_1 \otimes \cdots \otimes V_k$ *with a* k*-multilinear map* $\otimes$ *such that the following diagram commutes for all* k*-multilinear maps* f *and vector spaces* W*.*



As it turns out, tensor products are unique up to isomorphism, and thus it makes sense to call $V_1 \otimes \cdots \otimes V_k$ *the* tensor product of $V_1 \times \cdots \times V_k$. There is also no ambiguity in our notation $V_1 \otimes \cdots \otimes V_k$, as the taking the tensor product is an associative operation.

Theorem 1. *The tensor product is unique up to (unique) isomorphism. In particular, for all vector spaces* V *and* W*, there is a tensor product* $V \otimes W$*, and*

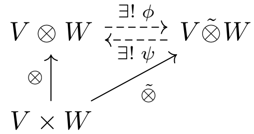
$V \tilde{\otimes} W$ *is also a tensor product* $\iff$



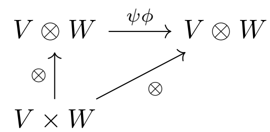
Furthermore, for any vector spaces A, B, C*, we have* $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$*.*

Proof. We will not prove existence here, as the construction is complicated. Instead, see Theorem 8 in these notes.

Uniqueness: Suppose $V \tilde{\otimes} W$ is also a tensor product, then the universal property gives the following commutative diagram.

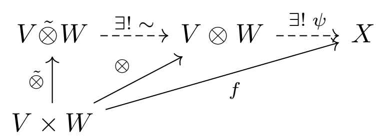


But this means we can form the following commutative diagram.



We know from the universal property that the extension of $\otimes$ must be unique, and id is certainly an extension, so $\psi\phi = \mathrm{id}$. Similarly, we can show $\phi\psi = \mathrm{id}$. Thus ϕ and ψ are isomorphisms, i.e. $V \otimes W \cong V \tilde{\otimes} W$.

Conversely, suppose the diagram from the statement of the theorem commutes, then the following diagram must also commute for any vector space X and bilinear $f : V \times W \to X$.

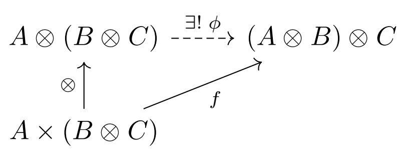


The composition along the top is then our desired linear map satisfying the universal property of the tensor product, so $V \tilde{\otimes} W$ is a tensor product.

Associativity: Consider the map

$$f : A \times (B \otimes C) \to (A \otimes B) \otimes C$$
$$(a, b \otimes c) \mapsto (a \otimes b) \otimes c.$$

This is bilinear since it's the same as $(a, b \otimes c) \mapsto \phi_a(b, c)$, where ϕ_a is the map from the universal property extending the bilinear map $f_a : (b, c) \mapsto (a \otimes b) \otimes c$. But then by the universal property, the following diagram commutes.



We can similarly construct a map $\psi : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, and these maps are mutually inverse. Thus $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.

□

Note that we can induct on this result, so this applies to any product of k vector spaces. The existence and uniqueness of the tensor product allow us to perform many useful algebraic constructions, but for our purposes, we will need only one. Consider two multilinear maps

$$V \xrightarrow{\phi} V', \qquad W \xrightarrow{\psi} W',$$

where both pairs (V, W) and (V', W') of vector spaces have the same base field. Then we can uniquely extend these two linear maps to a single linear map on $V \otimes W$.

Proposition 1. *Given multilinear maps $\phi : V \to V'$ and $\psi : W \to W'$, there is a unique linear map $V \otimes W \to V' \otimes W'$ mapping*

$$v \otimes w \mapsto \phi(v) \otimes \psi(w).$$

Proof. Consider the multilinear map $V \times W \to V \otimes W$ given by $(v, w) \mapsto \phi(v) \otimes \psi(w)$. By the universal property of the tensor product, this extends uniquely to a map $v \otimes w \mapsto \phi(v) \otimes \psi(w)$. □

This construction reduces quite nicely when working with tensors. Suppose S is a k-tensor on V and T is an ℓ-tensor on V. Then since $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}$ with $r \otimes s = rs$, we can apply the previous proposition to get a single linear map

$$\bigotimes_{i=i}^{k+\ell} V \to \mathbb{R}$$

$$v_1 \otimes \cdots \otimes v_{k+\ell} \mapsto S(v_1, \ldots, v_k)\, T(v_{k+1}, \ldots, v_{k+\ell}).$$

Pre-composing this with the tensor inclusion $\otimes$ then gives a multilinear map

$$S \otimes T : \prod_{i=1}^{k+\ell} V \to \mathbb{R}$$

$$(v_1, \ldots, v_{k+\ell}) \mapsto S(v_1, \ldots, v_k)\, T(v_{k+1}, \ldots, v_{k+\ell}).$$

Thus given a k-tensor and an ℓ-tensor on the same vector space, we can produce a $(k+\ell)$-tensor through this process. In the next section, we will use this product of tensors to define the wedge product of differential forms, which will be the last bit of theoretical foundation necessary to state and prove the generalized Stokes' Theorem.

4 DIFFERENTIAL FORMS

A tensor is **alternating** if swapping any two inputs negates the output. A differential form is then a way of assigning an alternating tensor to each point on a manifold. To start, we'll need a way of making any tensor alternating.

Proposition 2. *Given a k-tensor T, the tensor*

$$Alt(T) := \frac{1}{k!} \sum_{\sigma \in S_k} (sign\ \sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

is alternating.

Definition 3. *A* ***differential k-form*** *on a manifold M associates to each $p \in M$ an alternating k-tensor*

$$\omega_p : \bigoplus_{i=1}^{k} T_p(M) \to \mathbb{R}.$$

Suppose we're working on a chart $U \subset M$ with coordinates $(x_1, \ldots, x_n)$, then the differentials $dx_1, \ldots, dx_n$ are all 1-forms. In fact, we can combine these to form a basis for all differential forms on U up to degree n. This combination process is formalized by the wedge product.

Definition 4. *Suppose S is a k-tensor and T is an ℓ-tensor, then their* ***wedge product*** *is the $(k+\ell)$-tensor*

$$S \wedge T := \frac{(k+\ell)!}{k!\, \ell!} Alt(S \otimes T).$$

The wedge product satisfies several basic properties that are useful in computations. We list them here without proof:

- Associativity: $(R \wedge S) \wedge T = R \wedge (S \wedge T)$;
- Anticommutativity: $S \wedge T = -T \wedge S$, which implies $T \wedge T = 0$;
- Homogeneity: $(\lambda S) \wedge T = S \wedge (\lambda T) = \lambda(S \wedge T)$; and
- Distributivity: $(R + S) \wedge T = (R \wedge T) + (S \wedge T)$.

As promised, the wedge product allows us to construct any differential form on a local coordinate chart.

Theorem 2. *On a chart with coordinates $(x_1, \ldots, x_n)$, any k-form ω (for $k \leq n$) can be written uniquely as*

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k} \omega_{i_1,\ldots,i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where each $\omega_{i_1,\ldots,i_k}$ is a scalar function.

It is common to see statements like the one above written in *multi-index notation*, in which case it reads

$$\omega = \sum_I \omega_I \, dx_I.$$

Additionally, we will usually suppress the $\wedge$ when working with dx_i, since it should be clear from context that we are taking their wedge product.

To differentiate forms, we need a generalization of the derivative called the **exterior derivative**.

Definition 5. *Suppose we have a k-form $\omega = \sum_I \omega_I \, dx_I$, then its **exterior derivative** is the $k+1$-form*

$$d\omega := \sum_I d\omega_I \wedge dx_I,$$

where

$$d\omega_I = \sum_i \frac{\partial \omega_I}{\partial x_i} dx_i$$

is the usual differential of a scalar function.

The exterior derivative has the following useful properties:

- $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$;
- if ω_1 is a k-form, then $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$; and
- $d^2 = 0$.

Finally, we need a notion of integration of forms. To do this, we'll need to define partitions of unity.

Definition 6. *A **partition of unity** on M is a collection $\{\phi_i\}_i$ of continuous maps $\phi_i : M \to \mathbb{R}$ such that*

1. *$\phi_i \geq 0$ for all i;*

2. *every $p \in M$ has a neighborhood on which all but finitely many of the ϕ_i are 0;*

3. *Each ϕ_i is 0 except on some closed set contained in one of the charts U of M; and*

4. *For all $p \in M$, we have $\sum_i \phi_i(p) = 1$.*

Given a partition of unity of M, we can define what it means to integrate a differential form ω over M.

Definition 7. *Suppose M is a manifold with charts (U_i, ϕ_i), then take a partition of unity $\{\psi_i\}_i$. We define the intergral of ω over M as*

$$\int_M \omega := \sum_i \int_M \psi_i \omega.$$

Importantly, this integral does not depend on the choice of charts or the choice of the partition of unity.

5 STOKES' THEOREM

With the theory of manifolds and differential forms built up, we can finally state and prove the generalized Stokes' Theorem.

Theorem 3 (Stokes' Theorem)**.** *Let M be an oriented n-manifold, and let ω be an $(n-1)$-form with compact support on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. We will first prove the theorem when $M = \mathbb{H}^n$, the n-dimensional upper half-plane. Then we will extend the result to when M is a general manifold (potentially with boundary).

Since ω has compact support on M, we can find $R > 0$ such that $A := [-R, R] \times \cdots \times [-R, R] \times [0, R]$ contains $\operatorname{supp}(\omega)$ (strictly so in the first $n-1$ dimensions). Additionally, since ω is an $(n-1)$-form, we can write ω locally on any patch $U \subset M$ with coordinates $(x_1, \ldots, x_n)$ as

$$\omega = \sum_{i=1}^{n} \omega_i \; dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

for some maps $\{\omega_i : U \to \mathbb{R}\}_{i=1}^n$. Its exterior derivative is then

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} d\omega_i \; dx_1 \cdots \widehat{dx_i} \cdots dx_n \\ &= \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x_j} \; dx_j \; dx_1 \widehat{dx_i} \cdots dx_n. \end{aligned}$$

If $i \neq j$, then there are two copies of dx_j in the expression above, so it becomes 0. Thus the only nonzero terms in the sum are those where $i = j$. This then becomes

$$\begin{aligned} &= \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial x_i} \, dx_i \, dx_1 \cdots \widehat{dx_i} \cdots dx_n \\ &= \sum_{i=1}^{n} (-1)^{(i-1)} \frac{\partial \omega_i}{\partial x_i} dx_1 \cdots dx_n. \end{aligned}$$

Since ω is identically 0 on $\mathbb{H}^n - A$, we know $d\omega = 0$ on $\mathbb{H}^n - A$. Thus integrating $d\omega$ over $\mathbb{H}^n$ gives

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^{n} (-1)^{(i-1)} \int_A \frac{\partial \omega_i}{\partial x_i} \, dx_1 \cdots dx_n \\ &= \sum_{i=1}^{n} (-1)^{(i-1)} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x_i} \, dx_1 \cdots dx_n. \end{aligned}$$

We can simplify this expression further, though. Since the first $n-1$ dimensions of $\operatorname{supp}(\omega)$ are strictly contained in A, we have $\omega_i(x) = 0$ whenever any coordinate of x has absolute value at least R. Thus

$$\begin{aligned} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x_i} \, dx_1 \cdots dx_n &= \int_0^R \int_{-R}^R \cdots \int_{-R}^R \big[\omega_i\big]_{x_i=-R}^{x_i=R} \, dx_1 \cdots dx_{n-1} \\ &= \int_0^R \int_{-R}^R \cdots \int_{-R}^R 0 \, dx_1 \cdots dx_{n-1} \\ &= 0. \end{aligned} \qquad (\star)$$

We can then simplify $\int_{\mathbb{H}^n} d\omega$ to

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= (-1)^{(n-1)} \int_{-R}^R \cdots \int_{-R}^R \big[\omega_n(x)\big]_{x_n=0}^{x_n=R} \, dx_1 \cdots dx_{n-1} \\ &= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(x_1, \ldots, x_{n-1}, 0) \, dx_1 \cdots dx_{n-1}. \end{aligned}$$

This is the most we can simplify, so we can begin calculating $\int_{\partial\mathbb{H}^n} \omega$ to see if it matches this. We have

$$\int_{\partial\mathbb{H}^n} \omega = \sum_{i=1}^{n} \int_{A \,\cap\, \partial\mathbb{H}^n} \omega_i \, dx_1 \cdots \widehat{dx_i} \cdots dx_n.$$

Now on $\partial\mathbb{H}^n$, the n-th coordinate x_n is identically 0, so $dx_n = 0$. Thus the only nonzero term in the above sum is when $i = n$. This then becomes

$$\begin{aligned} &= \int_{A \,\cap\, \partial\mathbb{H}^n} \omega_n(x_1, \ldots, x_{n-1}, 0) \, dx_1 \cdots dx_{n-1} \\ &= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(x_1, \ldots, x_{n-1}, 0) \, dx_1 \cdots dx_{n-1} \\ &= \int_{\mathbb{H}^n} d\omega. \end{aligned}$$

Thus Stokes' Theorem holds in the special case $M = \mathbb{H}^n$. Now suppose $M = \mathbb{R}^n$. This case is much simpler, as we've already done most of the work. Recall the argument used in the $(\star)$ computation. We used a covering $[-R,R] \times \cdots \times [-R,R] \times [0,R]$ then, but since we're working in $\mathbb{R}^n$ now, we can use a covering of the form $[-R,R] \times \cdots \times [-R,R]$. Then the argument in $(\star)$ applies to all i, so

$$\int_{\mathbb{R}^n} d\omega \quad = \quad \sum_{i=1}^{n} (-1)^{(i-1)} \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial \omega_i}{\partial x_i} \, dx_1 \cdots dx_n \quad = \quad 0.$$

And since $\mathbb{R}^n$ has no boundary, $\int_{\partial \mathbb{R}^n} \omega = 0 = \int_{\mathbb{R}^n} d\omega$. These past two special cases show that the theorem holds for any neighborhood of any manifold. Now we must extend these results to the entirety of any manifold.

Since $\operatorname{supp}(\omega)$ is compact, we can find a finite open cover $\{(U_i, \phi_i)\}_i$ of it. Now take a partition of unity $\{\psi_i\}_i$. Then since we know the theorem holds on any neighborhood of M, we have

$$\begin{aligned}
\int_{\partial M} \omega &= \sum_i \int_{\partial M} \psi_i \omega \\
&= \sum_i \int_M d(\psi_i \omega) \\
&= \sum_i \int_M \left(d\psi_i \wedge \omega + \psi_i \, d\omega \right).
\end{aligned}$$

By the linearity of d and using the fact that $\sum_i \psi_i = 1$, this becomes

$$\begin{aligned}
&= \int_M d\left(\sum_i \psi_i \right) \wedge \omega + \int_M \left(\sum_i \psi_i \right) \, d\omega \\
&= \int_M 0 \wedge \omega + \int_M d\omega \\
&= \int_M d\omega.
\end{aligned}$$

Thus Stokes' Theorem holds for any orientable manifold. □

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